TOPOLOGY - III, SOLUTION SHEET 6

Exercise 1. (1) It follows from the definition of a deformation retract that the spaces A and X are homotopy equivalent via the maps $i : A \to X$ and $r := F(x, 1) : X \to A$. Therefore i_* and r_* define mutually inverse maps between $H_k(A)$ and $H_k(X)$.

(2) It follows immediately from the long exact sequence of relative homology and (1) that $H_k(X, A) = 0$ for all k.

(3) We have the following long exact sequence of a triple:

$$... \to H_i(B, A) \to H_i(X, A) \to H_i(X, B) \to H_{i-1}(B, A) \to ...$$

It follows from (1) that the groups $H_i(B, A)$ vanish for all i. This gives us the required isomorphisms $H_i(X, A) \to H_i(X, B)$.

(4) Saying that a pair (Y, B) is a retract of (X, A) just means that there is a map $r : X \to Y$ such that r is deformation retract of X onto Y and $r_{|_B} : B \to A$ is a deformation retraction onto A.

Now, we have the following long exact sequence associated to the triple $B \subseteq A \subseteq X$:

$$... \to H_i(Y, B) \to H_i(X, B) \to H_i(X, Y) \to H_{i-1}(Y, B) \to ...$$

It follows from (1) that the groups $H_i(X, Y)$ vanish and hence we have that $H_i(Y, B) \cong H_i(X, B)$ for all i. Then by (2), it follows that $H_i(X, B) \cong H_i(X, A)$

Exercise 2. Please refer to Corollary 2.14 in Hatcher's book, on page 114.

Exercise 3. This is the content of the exposition at the start of page 125 in Hatcher's book.

Exercise 4. (1) This follows immediately from the long exact sequence in reduced relative homology for the pair (SX, C_-X) and the fact that C_-X is contractible which implies that $\tilde{H}_i(C_-) = 0$ for all i.

(2) Just like (1), This follows from the long exact sequence in reduced relative homology for the pair (C_+X, X) and the fact that C_+X is contractible.

(3) Let p be the vertex of the cone C_-X then by excision we have that $\tilde{H}_i((SX, C_-X)) = \tilde{H}_i((SX - \{p\}, C_-X - \{p\}))$ for all i. Then note that the pair (C_+X, X) is a deformation retract of the pair $(SX - \{p\}, C_-X - \{p\})$. Hence it follows from part (4) of exercise 1 that $\tilde{H}_i((SX, C_-X)) = \tilde{H}_i((SX - \{p\}, C_-X - \{p\})) \cong \tilde{H}_i(C_+X, X)$. Then the required isomorphisms follow from parts (1) and (2).

Exercise 5. The key observation is that the vertices of an irreducible $k-$ simplex in a barycentric subdivision of Δ^n are themselves barycentres of some sub-simplex of Δ^n. Therefore, the bijection is given by the following association; To a length $k-$ sequence $\sigma_0 \subsetneq \sigma_1 \subsetneq ... \subsetneq \sigma_k$ of sub-simplices of Δ, we associate the simplex $[b_0, ..., b_k]$, where b_i is the barycentre of σ_i.